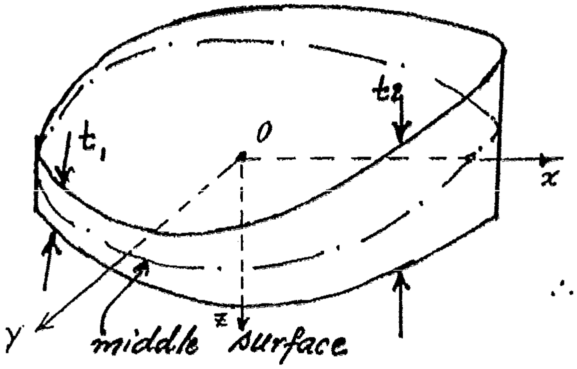
THEORY OF FLAT PLATES

1. INTRODUCTION:

PLATES are "flat" structural elements with relatively small dimension in one direction (thickness) Compared to other two dimensions.

DEFINITIONS, ASSUMPTIONS AND OBSERVATIONS:

MIDDLE SURFACE: Surface Containing all points that are at mid-depth across thickness.



(a) "Flat" Middle Surface
(b) Variable thickness plate
(c) Constant thickness plate.

Plates are weak in thickness direction
∴ Significant deflections occur in thickness direction

x, y, z Coordinate axes: In general, (x, y) axes are taken to be on the middle surface and z-direction is perpendicular downwards to the middle surface. (Right handed system).

Plates may be classified according to:

(a) Thickness Compared to Other dimensions → Thick Plate / Thin Plate / Extremely Thin Plate

(b) Deflections Compared to thickness → Small Deflections / Large Deflections

Here, our interests lie with "Thin Plates with Small Deflecti"

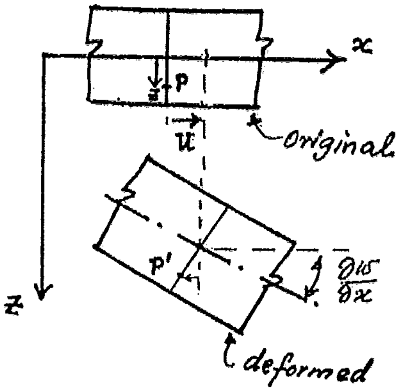
KIRCHHOFF ASSUMPTIONS:

> *1* Normals to the undeformed middle plane are assumed to remain straight, normal, and inextensional during the deformation, so that transverse normal and shearing strains may be neglected in deriving the plate kinematic relations.
>
> *2* Transverse normal stresses are assumed to be small compared with the other normal stress components, so that they may be neglected in the stress-strain relations.

∴ $\sigma_z = 0$; $\epsilon_z = 0$;

Since, $\epsilon_z = 0$ the z-direction deflection of every points on a normal is same as that of the middle - surface. $[w(x,y)]$

$$\therefore \; W(x,y,z) = w(x,y) \qquad \text{.... Eq. 1.1}$$



$P(x,y,z)$ deforms in x-z plane to P'

$$\therefore \; U(x,y,z) = u(x,y) - z \cdot \frac{\partial w}{\partial x} \qquad \text{.... Eq. 1.2}$$

Similarly

$$V(x,y,z) = v(x,y) - z \cdot \frac{\partial w}{\partial y} \qquad \text{... Eq. 1.3}$$

where, $u(x,y)$ and $v(x,y)$ are x, y displacements of the middle- surface.

2 THEORY

DISPLACEMENTS :

$$\left.\begin{aligned} U(x,y,z) &= u(x,y) - z\frac{\partial w}{\partial x} \\ V(x,y,z) &= v(x,y) - z\frac{\partial w}{\partial y} \\ W(x,y,z) &= w(x,y) \end{aligned}\right. \qquad \text{.... Eq. 2.1}$$

STRAIN - DISPLACEMENT RELATIONSHIPS: (Small deformations)

$$\left.\begin{aligned} \epsilon_{xx} &= \frac{dU}{dx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \\ \epsilon_{yy} &= \frac{\partial V}{\partial y} = \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} \\ \gamma_{xy} &= \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - 2z \cdot \frac{\partial^2 w}{\partial x \cdot \partial y} \end{aligned}\right. \qquad \text{.... Eq. 2.2}$$

$\epsilon_z = 0$; $\gamma_{xz} = 0$; $\gamma_{yz} = 0$ .... Assumed.

NOTE - In members subjected to small deflections the curvature

$$\mathcal{K}_x = \frac{1}{\rho_x} = -\frac{\partial^2 w}{\partial x^2} \; ; \; \mathcal{K}_y = \frac{1}{\rho_y} = -\frac{\partial^2 w}{\partial y^2} \; ; \; \mathcal{K}_{xy} = \frac{1}{\rho_{xy}} = -\frac{\partial^2 w}{\partial x \partial y}$$

$$\therefore \left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} + z\mathcal{K}_x \\ \epsilon_y &= \frac{\partial v}{\partial y} + z\mathcal{K}_y \\ \gamma_{xy} &= \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + 2z\mathcal{K}_{xy} \end{aligned}\right\}$$

also note that $\frac{\partial u}{\partial x}$; $\frac{\partial v}{\partial y}$ and $\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$ are middle-plane strains.

STRESS- STRAIN RELATIONSHIP: STRESS- DISPLACEMENT RELATIONS:

** Plate Material be homogeneous, continuous, isotropic, elastic and obeys Hooke's Law

assuming $\sigma_z = 0$; $\epsilon_z = 0$

$$\sigma_x = \frac{E}{(1-\nu^2)}\left[\epsilon_x + \nu\epsilon_y\right] = \frac{E}{(1-\nu^2)}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right] - \frac{Ez}{(1-\nu^2)}\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]$$

$$\sigma_y = \frac{E}{(1-\nu^2)}\left[\epsilon_y + \nu\epsilon_x\right] = \frac{E}{(1-\nu^2)}\left[\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right] - \frac{Ez}{(1-\nu^2)}\left[\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right]$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\cdot\gamma_{xy} = \frac{E}{2(1+\nu)}\cdot\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{Ez}{(1+\nu)}\cdot\frac{\partial^2 w}{\partial x\,\partial y}.$$

∴ The primary stresses are σ_x, σ_y and τ_{xy}.

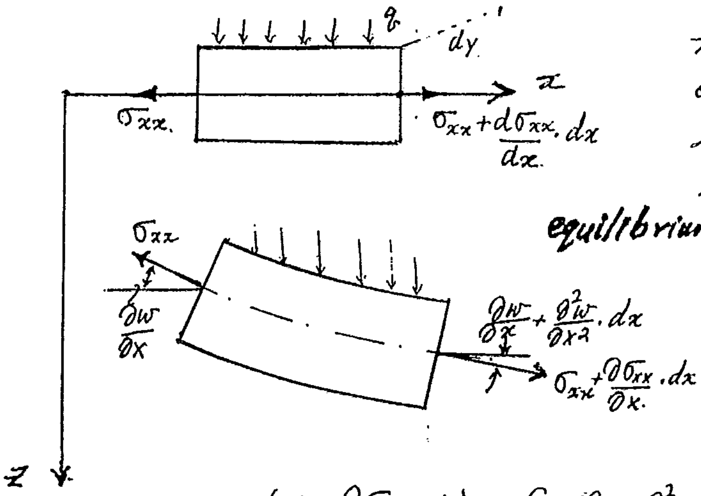
— - .. Eq. 2·3

Due to Kirchhoff's assumptions $\tau_{xz} = G\gamma_{xz} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = 0$; $\tau_{yz} = 0$

EQUILIBRIUM EQUATIONS:

NOTE: in the above discussions the strains and stresses are defined with respect to undeformed members and fixed Coordinate System (x, y, z). That means, σ_{xx} is the stress acting on the undeformed x-plane in x-direction The strains and stresses are referred to as Lagrangian strain and stress components or Green Strain Tensor and Kirchhoff Stress Tensor.

Since the plate undergoes significant deflection and rotations in z-direction, the stresses on an element of width "dy" in deformed state would be as shown in the figure. **Considering the equilibrium of original element in deformed state**

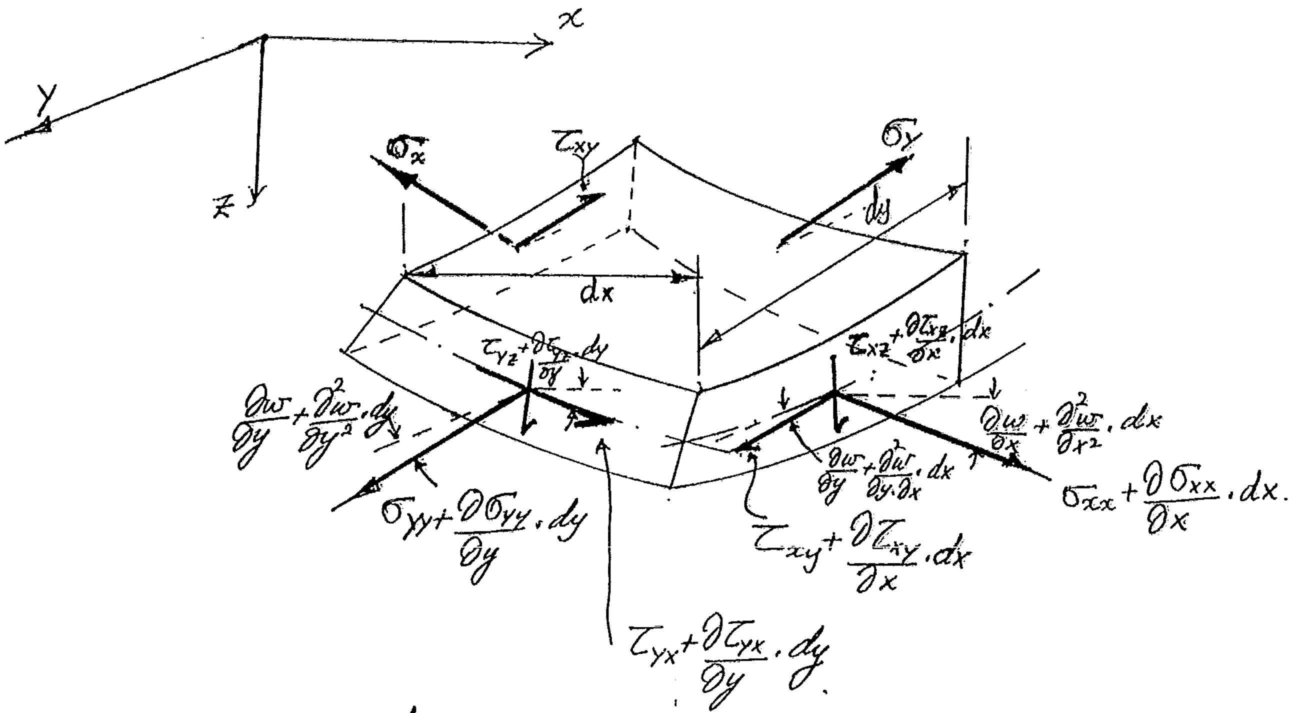

q
dy
x
σ_{xx}.
$\sigma_{xx} + \frac{d\sigma_{xx}}{dx}.dx$
σ_{xx}
$\frac{\partial w}{\partial x}$
$\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}.dx$
$\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}.dx$
z

Componant of Resulting force due to σ_{xx} in x-direction

$$\left(\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}.dx\right)dy.\cos\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}.dx\right) - \sigma_{xx}dy\cos\frac{\partial w}{\partial x}$$

$$= \frac{\partial\sigma_{xx}}{\partial x}.\,dx\,dy.$$

In z-direction $\left(\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}.dx\right)dy.\sin\left(\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}.dx\right) - \sigma_{xx}\cdot dy.\sin\frac{\partial w}{\partial x} = \left(\frac{\partial\sigma_{xx}}{\partial x}.\frac{\partial w}{\partial x} + \sigma_{xx}.\frac{\partial^2 w}{\partial x^2}\right)dx\,dy$



x
Y
Z
σ_x
τ_{xy}
σ_y
dy
dx
$\tau_{yz}+\frac{\partial\tau_{yz}}{\partial y}.dy$
$\tau_{xz}+\frac{\partial\tau_{xz}}{\partial x}.dx$
$\frac{\partial w}{\partial y}+\frac{\partial^2 w}{\partial y^2}.dy$
$\frac{\partial w}{\partial x}+\frac{\partial^2 w}{\partial x^2}.dx$
$\frac{\partial w}{\partial y}+\frac{\partial^2 w}{\partial y.\partial x}.dx$
$\sigma_{xx}+\frac{\partial\sigma_{xx}}{\partial x}.dx.$
$\sigma_{yy}+\frac{\partial\sigma_{yy}}{\partial y}.dy$
$\tau_{xy}+\frac{\partial\tau_{xy}}{\partial x}.dx$
$\tau_{yx}+\frac{\partial\tau_{yx}}{\partial y}.dy.$

Considering the plate rotations aswell, equilibrium equations

$$x \longrightarrow \qquad \frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+f_x=0$$

$$y \longrightarrow \qquad \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+f_y=0$$

$$z \rightarrow \qquad \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}+\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}\right)\frac{\partial w}{\partial x}$$

$$+\left(\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}\right).\frac{\partial w}{\partial y}+\left(\sigma_{xx}.\frac{\partial^2 w}{\partial x^2}+\tau_{xy}.\frac{\partial^2 w}{\partial x\partial y}+\tau_{yx}.\frac{\partial^2 w}{\partial x\partial y}+\sigma_{yy}\frac{\partial^2 w}{\partial y^2}\right)$$

$$+f_z=0 \qquad \text{---- Eq. 2.4.}$$

NOTE: that if the plate rotations are small then equations 2·4 reduce to $\sigma_{ij,j}+f_i=0$. Therefore, problems dealing with small displacements and small rotations can be developed either in un-deformed state or in deformed state.

NOTE: that in plate problems although $\frac{\partial w}{\partial x}$ ... small, if σ_x, σ_y & σ_{xy} are high then the product may be comparable to $\frac{\partial\tau_{xz}}{\partial x}$, etc.
∴ Eq. 2·4 has been used in the following general theory of Plates.

Eq.s 2.3 and 2.4 along with boundary conditions (both geometrical and statical) give solutions to displacement and stresses.

Unknowns; $u(x,y)$, $v(x,y)$, $w(x,y)$, $\sigma_{xx}(x,y,z)$, σ_{yy}, τ_{xy}, σ_{zz}, τ_{xz}, τ_{yz}.

From Kirchhoff assumptions σ_{zz}, γ_{xz} and $\gamma_{yz}=0 \therefore \sigma_{zz}, \tau_{xz}, \tau_{yz}=0$ or is it?

NOTE: the plates are elements primarily subjected z-direction loads on the surface. The z-direction loads are primarily transferred by stresses σ_{zz}, τ_{xz} and τ_{yz}. Therefore, although, σ_{zz}, τ_{xz} and τ_{yz} are small compared to other stresses it is necessary to maintain them in the equilibrium equation in order to satisfy statical boundary conditions.

NOTE: that since the plane-sections remain plane, the stress and strain distributions corresponding σ_{xx}, σ_{yy} and τ_{xy} are linear across the thickness. Thus, the distribution can be conveniently represented by stress Resultants acting at the mid-plane.

3. THEORY OF PLATES IN TERMS OF STRESS RESULTANTS

In Theory of Plates the following stress-resultants are used.

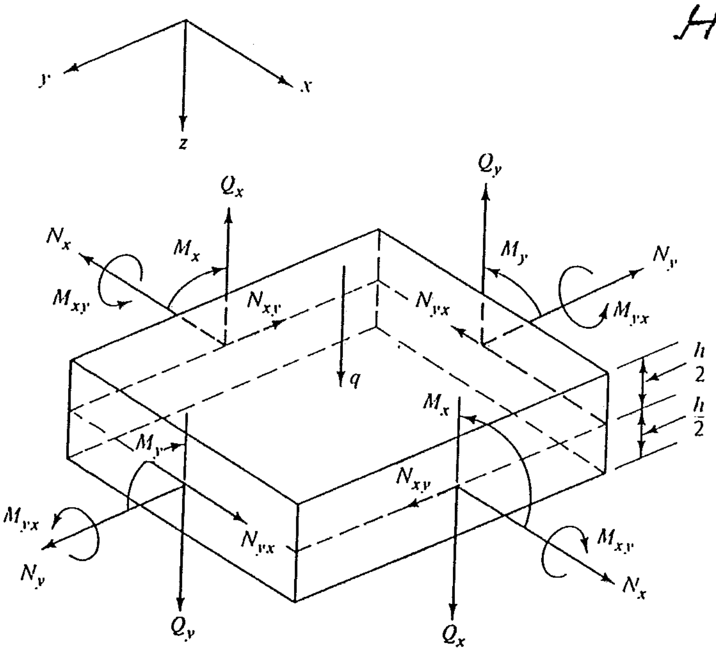


Figure. Senses of positive stress resultants and couples in rectangular cartesian coordinates.

Here;

$$N_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} \cdot dz \quad ; \quad N_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy} \cdot dz$$

$$N_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \cdot dz = N_{yx}$$

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} \cdot z \cdot dz \quad ; \quad M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{yy} \cdot z \cdot dz$$

$$M_{xy} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xy} \cdot z \cdot dz = M_{yx}$$

$$Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \cdot dz \quad ; \quad Q_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} \cdot dz$$

------ Eq. 3.1

NOTE: The positive directions of stress resultants are determined by positive stresses and positive 'z'.

NOTE: The resultants are defined per unit width of plate
$\therefore$ Units for $N_x, N_y, N_{xy}, Q_x, Q_y$ $[N/m]$
M_x, M_y, M_{xy} $[N \cdot m/m]$

STRESS - RESULTANTS — DISPLACEMENT RELATIONSHIPS.

Using Eq.s 3.1 and 2.3

$$N_x = \frac{Eh}{(1-\nu^2)} \cdot \left[\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y}\right] \quad ; \quad M_x = -D \cdot \left[\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right]$$

$$N_y = \frac{Eh}{(1-\nu^2)} \cdot \left[\frac{\partial v}{\partial y} + \nu \frac{\partial u}{\partial x}\right] \quad ; \quad M_y = -D \left[\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right]$$

$$N_{xy} = \frac{Eh}{2(1+\nu)} \cdot \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] \quad ; \quad M_{xy} = -D(1-\nu) \cdot \frac{\partial^2 w}{\partial x \partial y}.$$

------ Eq. 3.2

where, $D = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{E \cdot z^2}{(1-\nu^2)} \cdot dz = \frac{1}{12} \frac{Eh^3}{(1-\nu^2)}$ ~ Flexural Rigidity of a plate

NOTE: The stress resultants $N_x, N_y, N_{xy}, M_x, M_y, M_{xy}$ and Q_x, Q_y are defined at the Mid-plane and are functions of x and y only.

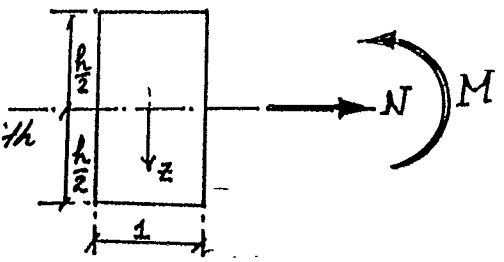
STRESS - STRESS RESULTANT RELATIONSHIP.

$$\sigma_x = \frac{N_x}{h} + \frac{M_x \cdot z}{(\frac{1}{12}h^3)}$$

$$\sigma_y = \frac{N_y}{h} + \frac{M_y \cdot z}{(\frac{1}{12}h^3)} \quad ;$$

$$\tau_{xy} = \frac{N_{xy}}{h} + \frac{M_{xy} \cdot z}{(\frac{1}{12}h^3)}$$

Compare with beam



-------- Eq. 3.3

EQUILIBRIUM EQUATIONS IN TERMS OF STRESS RESULTANTS

Integrating Eq. 2.4 through thickness

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{xx}}{\partial x}\, dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \tau_{yx}}{\partial y} \cdot dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \tau_{zx}}{\partial z} \cdot dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_x\, dz = 0$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} + \tau_{zx}\Big|_{-\frac{h}{2}}^{\frac{h}{2}} + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_x\, dz = 0$$

** Assuming No body forces and No surface shear stresses on surface $z = \pm h/2$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = 0. \qquad \ldots \text{Eq. } 3.4.1$$

Similarly, from the 2nd equation $\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0.$

$$\ldots \text{Eq. } 3.4.2$$

From the 3rd equation

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \sigma_{zz}\Big|_{-\frac{h}{2}}^{\frac{h}{2}} + \cancelto{0}{\left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y}\right)} \cdot \frac{\partial w}{\partial x} + \cancelto{0}{\left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}\right)} \cdot \frac{\partial w}{\partial y}$$

$$+ \left(N_x \cdot \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \cdot \frac{\partial^2 w}{\partial y^2}\right) = 0$$

Let the plate be loaded at surface $z = -\frac{h}{2}$ by a distributed load of 'q'. Therefore, $\sigma_{zz} = -q$ at $z = -\frac{h}{2}$ and $\sigma_{zz} = 0$ at $z = h/2$. In view of equations 3.4.1 and 3.4.2

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q + \left(N_x \cdot \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right) = 0$$

$$\ldots \text{Eq. } 3.4.3$$

Eq. 3.4.1, 3.4.2 and 3.4.3 are force equilibrium equations in x, y and z direction respectively.

Moment equilibrium equations can be derived by multiplying the equilibrium equations 2.4 by 'z' and then integrating through thickness.

$$\therefore \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{xx}}{\partial x} \cdot z \cdot dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \tau_{yx}}{\partial y} \cdot z \cdot dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \tau_{zx}}{\partial z} \cdot z \cdot dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_x \cdot z \cdot dz = 0.$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial (\tau_{zx} \cdot z)}{\partial z} dz - \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{zx} \cdot dz + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_x \cdot z \, dz = 0$$

Since, no body forces and surface forces at $z = \pm h/2$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} - Q_x = 0 \qquad \text{.... Eq. 3.4.4}$$

Similarly in y-direction

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \qquad \text{.... Eq. 3.4.5}$$

NOTE: Q_x and Q_y are the transverse shear force/unit length They can be related to displacements using equations 3.4.4, 3.4.5 and 3.2

$$\boxed{\begin{aligned} \therefore \quad Q_x &= -D \frac{\partial}{\partial x}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] \\ Q_y &= -D \frac{\partial}{\partial y}\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right] \end{aligned}} \qquad \text{... Eq. 3.5}$$

Infact Q_x and Q_y can be eliminated from equations 3.4 by differentiating eqs. 3.4.4 and 3.45 and substituting in eq. 3.4.3

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + q + \left(N_x \cdot \frac{\partial^2 w}{\partial x^2} + 2 N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \cdot \frac{\partial^2 w}{\partial y^2}\right) = 0$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \qquad \text{.... Eq. 3.6}$$

Equations 3.6 and stress-resultant - Displacement relations 3.2 constitute the governing equations of the plate in terms of N_x, N_{xy}, N_y, M_x, M_{xy}, M_y, u, v, and w.

Eq. 3.6 can be further reduced, using equations 3.2.

$$\boxed{\begin{aligned} \therefore \quad & \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \\ & \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \\ & D\left\{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right\} = q + \left(N_x \cdot \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \cdot \frac{\partial^2 w}{\partial x \partial y} + N_y \cdot \frac{\partial^2 w}{\partial y^2}\right) \end{aligned}}$$

.... Eq. 3.7.

Equs. 3.7 are in terms of N_x, N_{xy}, N_y and w. Using stress resultant - displacement relations between N_x, N_{xy}, N_y and u and v, the above equations can be written in terms of u, v and w only, and may be solved for u, v and w along with the boundary conditions.

Alternatively, in conjunction with the condition of Compatibility, the governing equations can be reduced to an eighth - order system of two equations in terms of transverse deflection w and a stress Function.

NOTE: In either case, the equations are coupled partial differential equations which may be solved by satisfying the appropriate geometrical and statical boundary Conditions.

The equations may be uncoupled and solved for deflections, stresses and strains fairly easily, under certain circumstances, which have been discussed below.

CASE:

A. Small deflection bending of thin plates subjected to combined lateral loads and In-plane forces in the middle surface.

NOTE: when a plate is subjected to In-plane forces $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ only, then in un-deformed state (plate remains flat)

$$N_x(x,y) = \bar{N}_x, \quad N_y(x,y) = \bar{N}_y, \quad N_{xy}(x,y) = \bar{N}_{xy}.$$

Now, if the plate is subjected to lateral loads and assuming that the lateral loads and accompanying small deflections do not change the in-plane stress distribution then the governing equation reduces to

$$\boxed{D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \cdot \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = q + \left[\bar{N}_x \cdot \frac{\partial^2 w}{\partial x^2} + 2\bar{N}_{xy} \cdot \frac{\partial^2 w}{\partial x \partial y} + \bar{N}_y \cdot \frac{\partial^2 w}{\partial y^2}\right]}$$

---- Eq. 3.8

CASE:

B. Small deflection bending of thin plates subjected to lateral loads only; then the governing equations reduce to

$$\boxed{D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \cdot \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] = q}$$

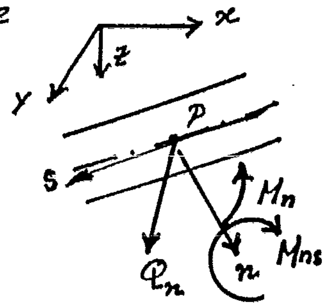
.... Eq. 3.9.

Equation 3.8 or 3.9 can be solved for w along with appropriate boundary conditions. Thus, the stress resultants, stresses and strains can be resolved. The relationships derived before are largely unaffected due to above assumptions. However, they should be viewed in light of the above assumptions.

4. BOUNDARY CONDITIONS

In the present plate theory (Thin plates small deflections) the governing equation (Eq. 3.8 or Eq. 3.9) is of the 4th order in x and 4th order in y. Therefore, it will be necessary to have eight boundary conditions or two B.C's on each edge of the plate.

The B.C's to be prescribed along each edge can be any combination of the following displacement and stresses;



Geometrical Condition — Statical Condition

$$w = \bar{w} \quad \text{or} \quad Q_n + \frac{\partial M_{ns}}{\partial s} = \bar{Q}_n + \frac{\partial \bar{M}_{ns}}{\partial s}$$

$$\frac{\partial w}{\partial n} = \overline{\frac{\partial w}{\partial n}} \quad \text{or} \quad M_n = \bar{M}_n . \qquad \ldots \text{Eq. } 4.1$$

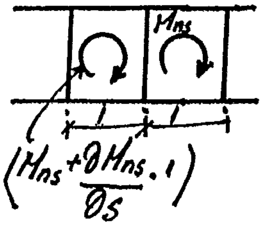
where, n and s are outward normal and tangental direction on middle-plane and barred quantities are prescribed.

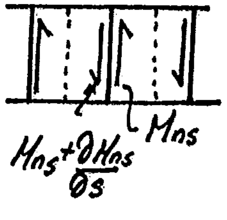
INCONSISTENCIES AT THE BOUNDARY:

At the boundary point 'P' there are three stress resultants M_n, M_{ns} and Q_n. If the assumption that normals to the undeformed middle plane remains normal had not been made the boundary Conditions would have included that either twist at point 'P' be zero or $M_{ns} = 0$. Due to Kirchhoff assumption when $w = 0$; it is implied that twist along the edge is zero. So, when $w \neq 0$, (twist $\neq 0$) then $Q_n = 0$ and $M_{ns} = 0$ which should really be one condition. This can be accomplished by establishing Kirchhoff shear resultant along the edges.

Assumption: Considering a unit width along the edge, the stresses due to M_{ns} can be degenerated by a couple made of forces equal to M_{ns} (see Figure).

$\therefore$ Unbalance force / unit length $= \frac{\partial M_{ns}}{\partial s}$.

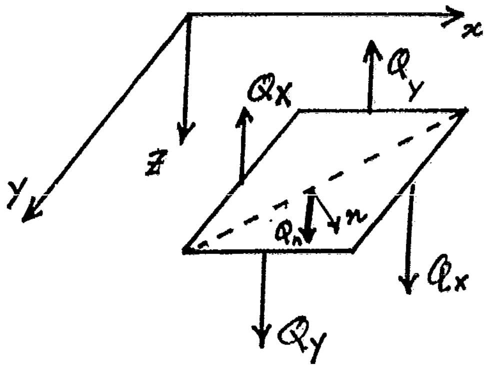

Mns
1
1
$\left(M_{ns} + \frac{\partial M_{ns}}{\partial s} \cdot 1\right)$

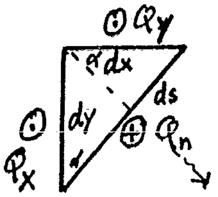


Therefore, Q_n and M_{ns} can be combined to form Kirchhoff shear resultant V_n

where $$V_n = Q_n + \frac{\partial M_{ns}}{\partial s} \quad \text{.... Eq. 4.2.}$$

SHEAR STRESS RESULTANT :



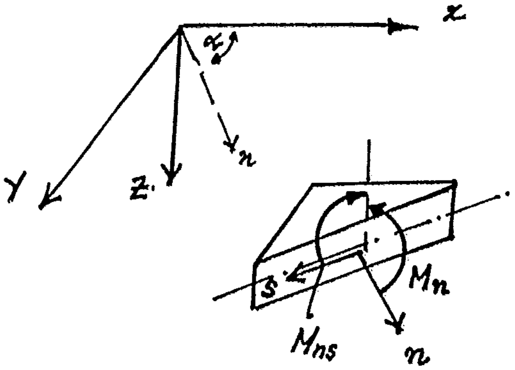


$$Q_n\, ds = Q_x \cdot dy + Q_y \cdot dx$$

$$\therefore \quad Q_n = Q_x \cdot \cos\alpha + Q_y \sin\alpha.$$

.... Eq. 4.3

MOMENT STRESS RESULTANTS:



$$M_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_n \cdot z \cdot dz \; ; \; M_{ns} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{ns} \cdot z \cdot dz$$

From stress transformation

$$\sigma_n = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{ns} = -\left(\frac{\sigma_x + \sigma_y}{2}\right) \sin 2\theta + \tau_{xy} \cos 2\theta$$

$$\sigma_s = \frac{\sigma_x + \sigma_y}{2} - \left(\frac{\sigma_x - \sigma_y}{2}\right) \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\therefore \quad M_n = \frac{M_x + M_y}{2} + \left(\frac{M_x - M_y}{2}\right) \cos 2\theta + M_{xy} \sin 2\theta.$$

$$M_{ns} = -\left(\frac{M_x - M_y}{2}\right) \sin 2\theta + M_{xy} \cos 2\theta. \quad \text{... Eq. 4.4}$$

$$M_s = \frac{M_x + M_y}{2} - \left(\frac{M_x - M_y}{2}\right) \cos 2\theta - M_{xy} \sin 2\theta$$

Built-in Edge (Fixed Edge, Clamped Edge)

$$w = 0 \quad ; \quad \frac{\partial w}{\partial n} = 0 \qquad \text{along the edge}$$

Free - Edge

$$M_n = 0 \quad ; \quad V_n = 0 \qquad \text{along the free edge}$$

$$\therefore \frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial s^2} = 0 \quad \text{and} \quad \frac{\partial^3 w}{\partial n^3} + (2-\nu)\cdot\frac{\partial^3 w}{\partial n \cdot \partial s^2} = 0 \qquad \text{along the free edge}$$

or

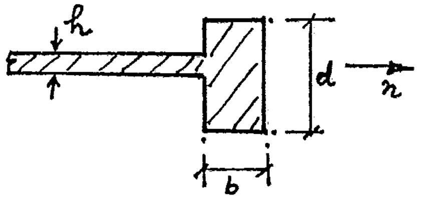
$$\frac{\partial}{\partial n}\left[\frac{\partial^2 w}{\partial n^2} + \frac{\partial^2 w}{\partial s^2}\right] = 0$$

SIMPLY SUPPORTED Edge

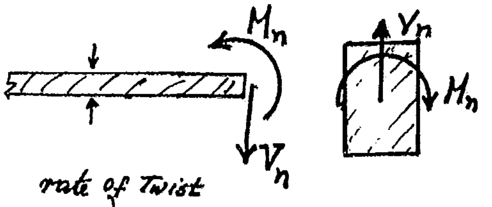
$$w = 0 \quad ; \quad M_n = 0 \qquad \text{along the simply Supported Edge}$$

$$w = 0 \quad ; \quad \frac{\partial^2 w}{\partial n^2} + \nu \overset{=0}{\cancel{\frac{\partial^2 w}{\partial s^2}}} = 0 \qquad \therefore \; w = 0 \text{ and } \frac{\partial^2 w}{\partial n^2} = 0$$

Elastically Supported and Elastically Built-in Edge



Say the plate is rigidly joined to a supporting beam, the deflection along the edge is not zero and is equal to the deflection of the beam. Also, the rotation is equal to the twist of the beam. Let, K_z and K_θ be the flexural and torsional rigidity of the beam



Considering beam deflection equation

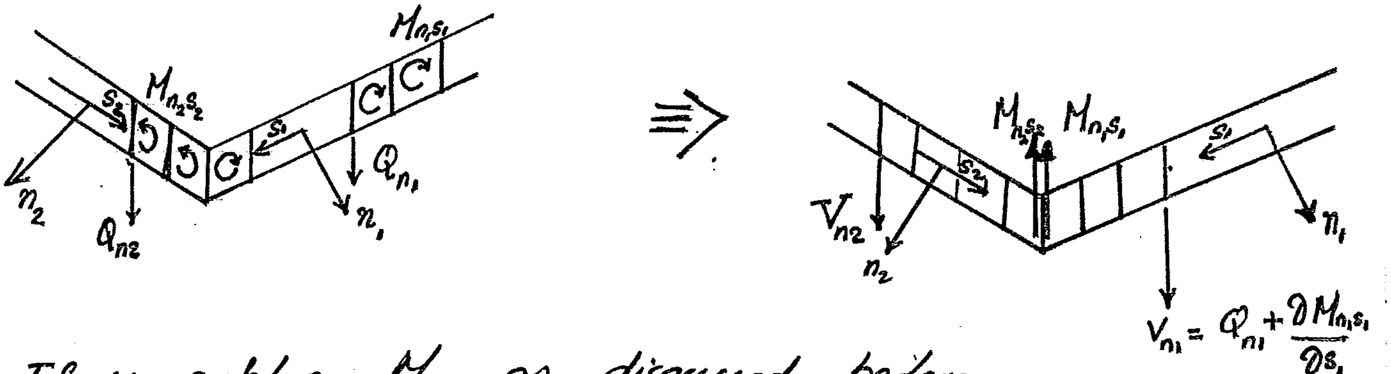
$$\left.\begin{aligned} K_z \cdot \frac{\partial^4 w}{\partial s^4} &= -V_n \\ K_\theta \cdot \frac{\partial^2 w}{\partial s \cdot \partial n} &= M_n \end{aligned}\right\} \text{along the edge}$$

$$K_\theta \cdot \overset{\substack{\text{rate of Twist} \\ \downarrow}}{\theta} = M_n$$

$$\therefore K_z \cdot \frac{\partial^4 w}{\partial s^4} = D\frac{\partial}{\partial n}\left[\frac{\partial^2 w}{\partial n^2} + (2-\nu)\frac{\partial^2 w}{\partial s^2}\right] ; \quad K_\theta \cdot \frac{\partial^2 w}{\partial s \cdot \partial n} = -D\left[\frac{\partial^2 w}{\partial n^2} + \nu \frac{\partial^2 w}{\partial s^2}\right]$$

at the edge

ANOMALIES AT THE SHARP CORNERS:



$M_{n_1s_1}$
$M_{n_2s_2}$
s_2
s_1
n_2
n_1
Q_{n2}
Q_{n1}
$\Rightarrow$
$M_{n_2s_2}$ $M_{n_1s_1}$
V_{n2}
$V_{n1} = Q_{n1} + \frac{\partial M_{n_1s_1}}{\partial s_1}$

If we replace M_{ns} as discussed before we not only obtain V_n along edges but at a sharp corner an unbalanced concentrated force

$$R = M_{n_1s_1} + M_{n_2s_2}$$

∴ To satisfy equilibrium of a plate with sharp edges it will be necessary to have internal forces M_n, V_n and concentrated load R at the sharp corners.

or A Plate with sharp corner loaded laterally will produce concentrated reaction at the sharp corner.

NOTE: that V_{n1}, V_{n2} and R are in opposite direction. In a rectangular plate $R = 2M_{xy}$ at the corner. ∴ A Flat plate loaded on the top will have a tendency to rise at the corners, to remain flat it has to be pulled down at the corners.

EXAMPLE: Show that the solution of a uniformly loaded elliptical plate with clamped edge can be taken as

$$w(x,y) = w_0\left[1 - \left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2\right]^2$$

where, 'a' and 'b' are the half-axis in x and y axis direction respectively.

2) Determine the magnitude of w_0 in terms of the load intensity q_0.

[Hint: deflection equation should satisfy all boundary conditions and governing equation $\nabla^4 w = q/D$]

METHODS OF SOLUTION

The differential equation (Eq. 3·8 or 3·9) is a partial differential equation of 4th order. Exact and approximate solutions to this equation can be obtained for thin elastic plates with various boundary conditions The solution is said to be exact in the sense that an infinite series solution can be truncated to obtain any desired degree of accuracy.

1. Double Fourier Series

The deflection w and load q are expressed interms of double fourier series, which may be double sine series, double Cosine - Sine series or double cosine series. Obviously, the function should satisfy all geometrical and statical boundary conditions.

$$\left.\begin{aligned} f_1(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \operatorname{Sin}\frac{m\pi x}{a} \cdot \operatorname{Sin}\frac{n\pi \cdot y}{b} \\ f_2(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn} \operatorname{Sin}\frac{m\pi x}{a} \cdot \operatorname{Cos}\frac{n\pi y}{b} \\ f_3(x,y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn} \operatorname{Cos}\frac{m\pi x}{a} \cdot \operatorname{Cos}\frac{n\pi}{b} y \end{aligned}\right\} \begin{aligned} &0 \leq x \leq a \\ &0 \leq y \leq b \end{aligned}$$

Also note that

$$\int_0^a \operatorname{Sin}\frac{m\pi x}{a} \cdot \operatorname{Sin}\frac{i\pi x}{a}\, dx = \begin{cases} 0 & \text{if } m \neq i \\ \frac{a}{2} & \text{if } m = i \end{cases}$$

$$\int_0^a \operatorname{Cos}\frac{m\pi x}{a} \cdot \operatorname{Cos}\frac{i\pi x}{a} \cdot dx = \begin{cases} 0 & \text{if } m \neq i \\ \frac{a}{2} & \text{if } m = i \end{cases}$$

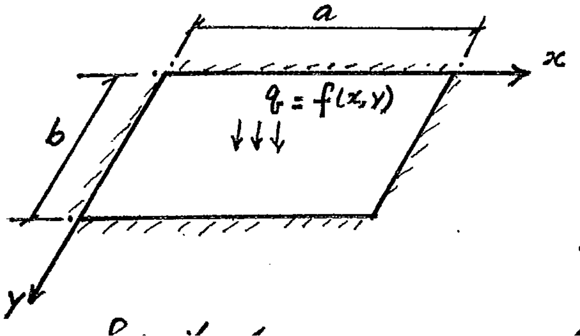
$$\int_0^a \operatorname{Sin}\frac{m\pi x}{a} \cdot \operatorname{Cos}\frac{i\pi x}{a} \cdot dx = \begin{cases} 0 & \text{if } m = i \text{ or } m \neq i \text{ with } |m \pm i| = \text{even} \\ \dfrac{2am}{\pi(m^2 - i^2)} & \text{if } m \neq i \text{ with } |m \pm i| = \text{odd} \end{cases}$$

a_{mn}, b_{mn} and C_{mn} are fourier Coefficients which can be obtained from

$$a_{ij} = \frac{4}{ab}\int_0^a\int_0^b f_1(x,y) \cdot \operatorname{Sin}\frac{i\pi x}{a} \cdot \operatorname{Sin}\frac{j\pi y}{b} \cdot dx \cdot dy$$

APPLICATIONS

SIMPLY SUPPORTED RECTANGULAR PLATES SUBJECTED TO ARBITRARY LOAD. (NAVIER SOLUTION)



All simply supported edges

at $x=0$ $w=0$; $M_x = 0$ $\therefore \frac{\partial^2 w}{\partial x^2} = 0$

at $x=a$ $w=0$; $\frac{\partial^2 w}{\partial x^2} = 0$

Similarly at $y=0$ $w=0$; $\frac{\partial^2 w}{\partial y^2} = 0$ ....Eq.

at $y=b$ $w=0$; $\frac{\partial^2 w}{\partial y^2} = 0$

The governing equation

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{f(x,y)}{D} \qquad \text{....Eq.}$$

Choose a function to satisfy the boundary condition. Show that

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \cdot \sin\frac{m\pi}{a}x \cdot \sin\frac{n\pi}{b}y \; ; \quad \begin{matrix} 0 \le x \le a \\ 0 \le y \le b \end{matrix}$$

Satisfies all eight boundary conditions.
Substituting in the governing equation

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \cdot \left[\left(\frac{m\pi}{a}\right)^4 + 2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 + \left(\frac{n\pi}{b}\right)^4\right] \cdot \sin\frac{m\pi}{a}x \cdot \sin\frac{n\pi}{b}y = \frac{f(x,y)}{D}$$

In fact the load $f(x,y)$ may be represented by a double fourier series as well

$$\therefore f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn} \cdot \sin\frac{m\pi}{a}x \cdot \sin\frac{n\pi}{b}y \; ; \quad \begin{matrix} 0 \le x \le a \\ 0 \le y \le b \end{matrix}$$

where,
$$f_{mn} = \frac{4}{ab} \cdot \int_0^a \int_0^b f(x,y) \cdot \sin\frac{m\pi}{a}x \cdot \sin\frac{n\pi}{b}y \cdot dx \cdot dy$$

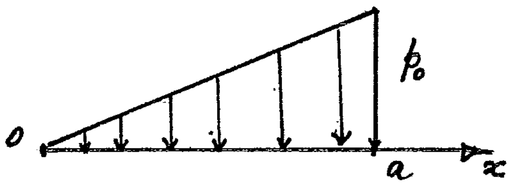
Re arranging

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{A_{mn}\left[\left(\frac{m\pi}{a}\right)^4+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4\right]-\frac{f_{mn}}{D}\right\}\sin\frac{m\pi x}{a}\cdot\sin\frac{n\pi y}{b}=0$$

In order the above equation may be satisfied at all points then,

$$A_{mn}=\frac{f_{mn}}{D\left[\left(\frac{m\pi}{a}\right)^4+2\left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2+\left(\frac{n\pi}{b}\right)^4\right]}.$$

(a) Plate under hydro-static pressure as shown.



$\therefore$ $x=0$ top-surface
$x=a$ bottom surface

$\therefore q=f(x,y)=p_0\cdot\frac{x}{a}$

$$\therefore f_{mn}=\frac{4}{ab}\cdot\int_0^a\int_0^b p_0\cdot\frac{x}{a}\cdot\sin\frac{m\pi}{a}x\,.\ \sin\frac{n\pi}{b}y\,.\ dx\cdot dy$$

$$=\frac{4p_0}{ab}\left[\left(\frac{x}{a}\right)\cdot\frac{\cos\frac{m\pi}{a}x}{\frac{m\pi}{a}}-\frac{\sin\frac{m\pi}{a}x}{\left(\frac{m\pi}{a}\right)^2}\right]_0^a\left[\frac{\cos\frac{n\pi}{b}y}{\left(\frac{n\pi}{b}\right)}\right]_0^b$$

$$f_{mn}=\frac{8p_0}{\pi^2 mn}(-1)^{m+1}\qquad\text{if } n \text{ is odd}$$

$$=0\qquad\text{if } n \text{ is even}$$

$\therefore$ DEFLECTION

$$\therefore w=\frac{8p_0}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1,3,5,\dots}^{\infty}\frac{(-1)^{m+1}}{mn\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\cdot\sin\frac{m\pi}{a}x\,.\ \sin\frac{n\pi}{b}y\,.$$

note that even terms in y-direction should vanish because they produce unsymmetric terms. For the problem under consideration deflection along y-direction is symmetric with respect to plate centerlines.

MOMENT RESULTANTS:

$$M_x = -D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right] = \frac{8p_0}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1,3,5\cdot\cdot}^{\infty}\frac{(-1)^m\left[\left(\frac{m}{a}\right)^2+\nu\left(\frac{n}{b}\right)^2\right]}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2_{mn}} \sin\frac{m\pi}{a}x \cdot \sin\frac{n\pi y}{b}$$

$$M_y = \frac{8p_0}{\pi^4}\cdot\sum_{m=1}^{\infty}\sum_{n=1,3,5\cdot\cdot}^{\infty}\frac{(-1)^m\left[\nu\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]}{mn\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2} \cdot \sin\frac{m\pi}{a}x \cdot \sin\frac{n\pi}{b}y$$

$$M_{xy} = \frac{8p_0(1-\nu)}{\pi^4 ab}\sum_{m=1}^{\infty}\sum_{n=1,3,5\cdot\cdot}^{\infty}\frac{(-1)^m\cdot}{\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2\right]^2}\cdot \cos\frac{m\pi}{a}\cdot x \cdot \cos\frac{n\pi}{b}y.$$

Similarly, the transverse shear force Q_x, Q_y, Kirchhoff stress resultants V_x, V_y and stresses σ_x, σ_y and τ_{xy} can be obtained in terms of a series.

If the values of 'p_0', 'a' and 'b' are given then the exact values of the above parameters can be obtained at any point (x, y).

For a Square Plate $a = b$

$$w = \frac{8p_0 a^4}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{m+1}}{mn[m^2+n^2]^2} \cdot \sin\frac{m\pi x}{a} \cdot \sin\frac{n\pi y}{a}$$

Deflection at the center of the plate $\therefore x = \frac{a}{2}$; $y = \frac{a}{2}$

$$\therefore w_c = \frac{8p_0 a^4}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{m+1}}{mn[m^2+n^2]^2}$$

$$= \frac{p_0 a^4}{D}\cdot\frac{8}{\pi^6}\left[\frac{1}{4} + \frac{1}{3[10]^2} + \frac{(-1)^3}{2[5]^2} + \frac{(-1)^3}{2\times 3[13]^2} + \frac{(-1)^4}{3\cdot[10]^2} + \ldots\right]$$

$$\therefore W\left(x = \frac{a}{2}, y = \frac{a}{2}\right) = 0.00196\,\frac{p_0 a^4}{D}$$

Maximum deflection will be along the line $y = \frac{a}{2}$

$$w\Big/_{y=\frac{b}{2}} = \frac{p_0 a^4}{D}\cdot\frac{8}{\pi^6}\sum_{m=1}^{\infty}\sum_{n=1,3,5}^{\infty}\frac{(-1)^{m+1}}{mn(m^2+n^2)^2}\cdot \sin\frac{m\pi x}{a}$$

$$w\Big/_{y=\frac{b}{2}} = \frac{p_0 a^4}{D}\cdot\frac{8}{\pi^6}\left[\left[\frac{1}{4}+\frac{1}{3\cdot(10)^2}+\frac{1}{5(26)^2}\right]\sin\frac{\pi x}{a} - \left[\frac{1}{2(5)^2}+\frac{1}{2\times3(13)^2}+\cdots\right]\right.$$

$$\sin\frac{2\pi x}{a} + \cdots\cdots \quad \sin\frac{3\pi x}{a}$$

$$\therefore w\Big/_{y=\frac{b}{2}} = \frac{p_0 a^4}{D}\cdot\frac{8}{\pi^6}\left[0.2533 \sin\frac{\pi x}{a} - 0.02111 \sin\frac{2\pi x}{a} + 0.0037 \sin\frac{3\pi x}{a}\right]$$

For maximum deflection $\frac{\partial w}{\partial x} = 0$

From the equation it can be shown that the maximum deflection occurs at $x = 0.557\ a$

$$\therefore \qquad w_{max} = 0.00206\ p_0\cdot\frac{a^4}{D}$$

AXISYMMETRIC CIRCULAR PLATES

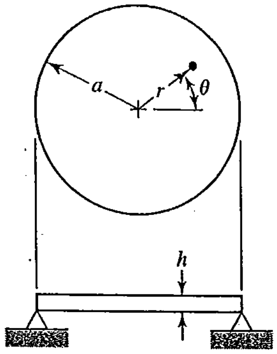


FIGURE 13.11 Simply supported circular plate.

Boresi, Arthur P., and Schmidt R., Advanced Mechanics of Materials, John Wiley & Sons, 2003.

$$\nabla^2\nabla^2 w = \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = \frac{p}{D} \tag{13.71}$$

The general solution of Eq. 13.71 is presented by Marguerre and Woernle (1969). Here, we consider only the axisymmetric case, in which the plate is loaded and supported symmetrically with respect to the z axis. Then, Eq. 13.71 reduces to (since dependency on θ vanishes)

$$\nabla^2\nabla^2 w = \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right) = \frac{p}{D} \tag{13.72}$$

The solution of Eq. 13.72, with $p = p_0 =$ constant, is

$$w = \frac{p_0 r^4}{64D} + A_1 + A_2 \ln r + B_1 r^2 + B_2 r^2 \ln r \tag{13.73}$$

where A_1, A_2, B_1, and B_2 are constants of integration. The constants A_1, A_2, B_1, and B_2 are determined by the boundary conditions at $r = a$ and the regularity conditions that w, ω_r (see Eq. 13.13), M_{rr}, and V_r must be finite at the center of the plate [origin $r = 0$ of the (r, θ) coordinate system].

Analogous to the expressions for the rectangular plate, we have (Eqs. 13.13, 13.28, 13.30 with $\alpha = 1$, $\beta = r$; see also Eq. 13.51), in general,

$$\begin{aligned}
M_{rr} &= -D\left[w_{rr} + \nu\left(\frac{w_r}{r}+\frac{w_{\theta\theta}}{r^2}\right)\right]\\
M_{\theta\theta} &= -D\left[\frac{w_r}{r}+\frac{w_{\theta\theta}}{r^2}+\nu w_{rr}\right]\\
M_{rr}+M_{\theta\theta} &= -D(1+\nu)\nabla^2 w\\
M_{r\theta} &= -D(1-\nu)\frac{\partial}{\partial r}\left(\frac{w_\theta}{r}\right)\\
V_r &= -D\left[\frac{\partial}{\partial r}(\nabla^2 w) + (1-\nu)\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{w_{\theta\theta}}{r}\right)\right]
\end{aligned} \tag{13.74}$$

$$V_\theta = -D\left[\frac{1}{r}\frac{\partial}{\partial\theta}(\nabla^2 w) + (1-\nu)\frac{\partial^2}{\partial r^2}\left(\frac{w_\theta}{r}\right)\right]$$

$$\omega_r = \frac{1}{r}w_\theta,\quad \omega_\theta = -w_r \qquad \text{(13.74) continued}$$

where subscripts (r, θ) on w denote partial differentiation. Accordingly, for the solid plate, by Eqs. 13.73 and 13.74, we conclude that $A_2 = B_2 = 0$ for axisymmetric conditions.

13.9.2 Circular Plates with Simply Supported Edges

For a solid circular plate simply supported at the edge $r = a$, the boundary conditions are, with Eqs. 13.73 and 13.74 and $A_2 = B_2 = 0$,

$$w(a) = A_1 + B_1 a^2 + \frac{p_0 a^4}{64D} = 0$$

$$-\frac{1}{D}M_{rr}(a) = 2(1+\nu)B_1 + (3+\nu)\frac{p_0 a^2}{16D} = 0$$

Hence, solving these equations for A_1 and B_1, we obtain with Eqs. 13.73 and 13.74 the following results for the simply supported solid circular plate with uniform lateral pressure $p = p_0$:

$$\begin{aligned}
w &= \frac{p_0 a^4}{64D}\left[1-\left(\frac{r}{a}\right)^2\right]\left[\frac{5+\nu}{1+\nu}-\left(\frac{r}{a}\right)^2\right] \\
M_{rr} &= \frac{p_0 a^2}{16}(3+\nu)\left[1-\left(\frac{r}{a}\right)^2\right] \qquad \text{(13.75)} \\
M_{\theta\theta} &= \frac{p_0 a^2}{16}\left[3+\nu-(1+3\nu)\left(\frac{r}{a}\right)^2\right]
\end{aligned}$$

13.9.3 Circular Plates with Fixed Edges

For a solid circular plate with fixed edge at $r = a$, the boundary conditions with $A_2 = B_2 = 0$ and Eqs. 13.73 and 13.74 are

$$w(a) = A_1 + B_1 a^2 + \frac{p_0 a^4}{64D} = 0$$

Rotation $$\omega_\theta(a) = -w_r(a) = -2B_1 a - \frac{p_0 a^3}{15D} = 0$$

Solving these equations for A_1 and B_1, we obtain by Eqs. 13.73 and 13.74 the following results for the solid circular plate with fixed edge at $r = a$, subject to uniform lateral pressure $p = p_0$:

$$w = \frac{p_0 a^4}{64D}\left[1-\left(\frac{r}{a}\right)^2\right]^2$$

$$M_{rr} = \frac{p_0 a^2}{16}\left[1+\nu-(3+\nu)\left(\frac{r}{a}\right)^2\right] \tag{13.76}$$

$$M_{\theta\theta} = \frac{p_0 a^2}{16}\left[1+\nu-(1+3\nu)\left(\frac{r}{a}\right)^2\right]$$

Equations 13.73–13.76 summarize the bending theory of simply supported and clamped circular plates subject to uniform lateral pressure. Numerous solutions for other types of plates, loadings, and boundary conditions have been presented by Marguerre and Woernle (1969). In particular, Marguerre and Woernle have presented extensive results for orthotropic plates.

13.9.4 Circular Plate with a Circular Hole at the Center

For a simply supported circular plate of radius a with circular hole of radius b at the center and subjected to uniform lateral pressure $p = p_0$, the boundary conditions are (see Eqs. 13.73 and 13.74)

$$\begin{aligned} V_r(b) &= -D\left(\frac{4B_2}{b}+\frac{p_0 b}{2D}\right) = 0 \\ M_{rr}(b) &= -D\left\{-(1-\nu)\frac{A_2}{b^2}+2B_1(1+\nu)\right. \\ &\qquad \left. + B_2[3+\nu+2(1+\nu)\ln b] + \frac{(3+\nu)p_0 b^2}{16D}\right\} = 0 \end{aligned} \tag{13.77}$$

and

$$\begin{aligned} w(a) &= A_1 + A_2\ln a + B_1 a^2 + B_2 a^2 \ln a + \frac{p_0 a^4}{64D} = 0 \\ M_{rr}(a) &= -D\left\{-(1-\nu)\frac{A_2}{a^2}+2B_1(1+\nu)\right. \\ &\qquad \left. + B_2[3+\nu+2(1+\nu)\ln a] + \frac{(3+\nu)p_0 a^2}{16D}\right\} = 0 \end{aligned} \tag{13.78}$$

Solving these equations for A_1, A_2, B_1, and B_2, we obtain

$$A_1 = -\frac{p_0 a^4}{4D}\left\{\frac{(1+\nu)\ln\frac{a}{b}\ln a}{(1-\nu)\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(5-\nu)\ln a}{4(1-\nu)\left(\frac{a}{b}\right)^2}\right.$$

$$\left. + \frac{\left(\frac{a}{b}\right)^2\ln a - \ln b}{2\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(3+\nu)\left[\left(\frac{a}{b}\right)^2-1\right]}{8(1+\nu)\left(\frac{a}{b}\right)^2} + \frac{1}{16}\right\}$$

$$A_2 = \frac{p_0 a^4}{4D}\left\{\frac{(1+\nu)\ln\frac{a}{b}}{(1-\nu)\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(3+\nu)}{4(1-\nu)\left(\frac{a}{b}\right)^2}\right\} \tag{13.79}$$

$$B_1 = \frac{p_0 a^2}{8D}\left\{\frac{\left(\frac{a}{b}\right)^2\ln a - \ln b}{\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(3+\nu)\left[\left(\frac{a}{b}\right)^2-1\right]}{4(1+\nu)\left(\frac{a}{b}\right)^2}\right\}$$

$$B_2 = -\frac{p_0 b^2}{8D}$$

With these coefficients and Eqs. 13.73 and 13.74, the displacement and stress resultants may be computed.

For example, for $a/b = 2$ and $\nu = 0.30$, the maximum displacement is

$$w_{\max} = w(b) = 0.682\,\frac{p_0 a^2}{Eh^3} \tag{13.80}$$

Except for simple types of loading and shapes of plates, such as a circular shape, the method of finding the bending moment by solving the plate equation (Eq. 13.56) is somewhat complicated. However, the results obtained can be reduced to tables or curves of coefficients for the maximum bending moments per unit width of a plate and for the maximum deflections of the plate. Some of these results are presented below.

The bending theory of elastic plates, however, does not make allowance for adjustments that take place when slight local yielding at portions of high stress causes a redistribution of stress. This redistribution of stress, in turn, may result in additional strength of the plate, which may often be incorporated into the design of plates, particularly plates of ductile material. We also observe that the bending theory of plates based on Eq. 13.56 does not take into account the added resistance of the plate resulting from direct tensile stresses that accompany relatively large deflections.

13.9.5 Summary for Circular Plates with Simply Supported Edges

Consider a circular plate with simply supported edges, so that no displacement occurs at the edge. The lateral displacement w and bending moments M_{rr}, $M_{\theta\theta}$ for uniform lateral pressure p are given by Eqs. 13.75. The maximum displacement occurs at the center of the plate ($r = 0$). The maximum stress $\sigma_{\max}$ also occurs at the center of the plate. The value of $\sigma_{\max}$ is tabulated in Table 13.2. Results are given in Table 13.2 also for the case of a spot load ($P = \pi r_0^2 p$) at the center of the plate, where the solution is reasonably accurate, provided r_0 is a sufficiently small (nonzero) value.

13.9.6 Summary for Circular Plates with Fixed Edges

Consider a circular plate rigidly held (fixed) so that no rotation or displacement occurs at the edge. We observe that under service conditions the edges of plates are seldom completely "fixed," although usually they are subject to some restraint; furthermore, a slight amount of yielding at the fixed edge may destroy much of the effect of the restraint and thereby transfer the moment to the central part of the plate. For these reasons, the restraint at the edges of a plate is considered of less importance than would be indicated by the results of the theory of flexure of plates with fixed edges, particularly if the plate is made of relatively ductile material. In general, an actual medium-thick plate with a fixed edge will be intermediate in stiffness between the plate with a simply supported edge and the plate with an ideally fixed edge.

Formulas are given in Table 13.2 for the maximum deflection of clamped circular plates of an ideal, elastic material (Morley, 1935). Experiments have verified the formulas for uniformly distributed loads and a simply supported edge. These experiments with fixed-edged plates under uniformly distributed loads show that the formula for the deflection is correct for thin and medium-thick plates [$(h/a) < 0.1$] for deflections not

TABLE 13.2 Formulas for Values of the Maximum Principal Stresses and Maximum Deflections in Circular Plates as Obtained by Theory of Flexure of Plates[a]

Support and loading	Principal stress ($\sigma_{\max}$)	Point of maximum stress	Maximum deflection ($w_{\max}$)
Edge simply supported; load uniform ($r_0 = a$)	$\frac{3}{8}(3+\nu)\,p\frac{a^2}{h^2}$	Center	$\frac{3}{16}(1-\nu)(5+\nu)\frac{pa^4}{Eh^3}$
Edge fixed; load uniform ($r_0 = a$)	$\frac{3}{4}\,p\frac{a^2}{h^2}$	Edge[b]	$\frac{3}{16}(1-\nu^2)\frac{pa^4}{Eh^3}$
Edge simply supported; load at center. $P = \pi r_0^2 p$, $r_0 \to 0$, but $r_0 > 0$	$\frac{3(1+\nu)}{2\pi h^2}P\left(\frac{1}{\nu+1} + \ln\frac{a}{r_0} - \frac{1-\nu}{1+\nu}\frac{r_0^2}{4a^2}\right)$	Center	$\frac{3(1-\nu)(3+\nu)Pa^2}{4\pi Eh^3}$
Fixed edge; load at center. $P = \pi r_0^2 p$, $r_0 \to 0$, but $r_0 > 0$	$\frac{3(1+\nu)}{2\pi h^2}P\left(\ln\frac{a}{r_0} + \frac{r_0^2}{4a^2}\right)$ a must be $> 1.7r_0$	Center	$\frac{3(1-\nu^2)Pa^2}{4\pi Eh^3}$

---

[a] a = radius of plate; r_0 = radius of central loaded area; h = thickness of plate; p = uniform load per unit area; ν = Poisson's ratio.

[b] For thicker plates ($h/r > 0.1$), the deflection is $w_{\max} = C\left(\frac{3}{16}\right)(1-\nu^2)(pa^4/Eh^3)$, where the constant C depends on the ratio h/a as follows: $C = 1 + 5.72(h/a)^2$.

larger than about one-half the plate thickness. For thicker plates the measured values of deflection are much larger than those computed by the formula. Two reasons for this discrepancy exist: 1. lack of ideal fixity at the edge and 2. additional deflection in the thicker plates resulting from the shear stresses. These experiments suggested that, for thicker circular plates $[(h/a) > 0.1]$ with fixed edges subjected to uniform loads, the values of $w_{\max}$ given in Table 13.2 should be multiplied by a factor that depends on the ratio of the thickness h to the radius r. This factor is $C = 1 + 5.72(h/a)^2$. Experiments on plates with edges securely clamped gave deflections that agreed closely with values computed by the use of the bending theory formula and the constant C.

Formulas for deflections by the bending theory give values that are too large for thin to medium-thick plates when the deflections are larger than about one-half the plate thickness.

13.9.7 Summary for Stresses and Deflections in Flat Circular Plates with Central Holes

Circular plates of radius a with circular holes of radius r_0 at their center are commonly used in engineering systems. For example, they occur in thrust-bearing plates, telephone and loudspeaker diaphragms, steam turbines, diffusers, piston heads, etc. Several cases of practical importance have been studied by Wahl and Lobo (1930). In all these cases, the maximum stress is given by simple formulas of the type

$$\sigma_{\max} = k_1 \frac{pa^2}{h^2} \quad \text{or} \quad \sigma_{\max} = \frac{k_1 P}{h^2} \tag{13.81}$$

depending on whether the applied load is uniformly distributed over the plate or concentrated along the edge of the central hole. Likewise, the maximum deflections are given by simple formulas of the type

$$w_{\max} = k_2 \frac{pa^4}{Eh^3} \quad \text{or} \quad w_{\max} = k_2 \frac{Pa^2}{Eh^3} \tag{13.82}$$

Wahl and Lobo have calculated numerical values for k_1 and k_2 for several values of the ratio a/r_0 and for a Poisson's ratio of $\nu = 0.30$. The cases that they studied are shown in Figure 13.12 and the corresponding values of k_1 and k_2 are tabulated in Table 13.3. For other solutions for symmetrical bending of circular plates, the interested reader is referred to Timoshenko and Woinowsky-Krieger (1959).

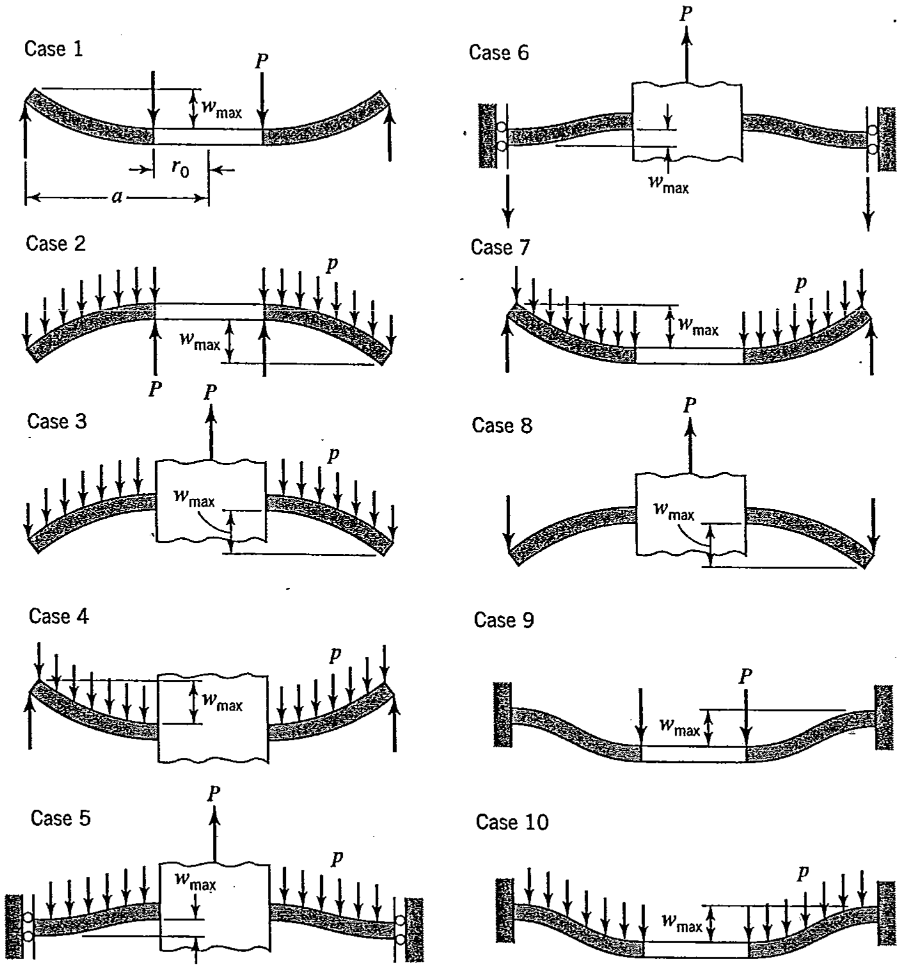


FIGURE 13.12 Circular plates with central holes, various loadings and boundary conditions.

TABLE 13.3 Coefficients k_1 and k_2 (Eqs. 13.81 and 13.82) for the Ten Cases Shown in Figure 13.12 [a]

	$\frac{a}{r_0} = 1.25$		$\frac{a}{r_0} = 1.5$		$\frac{a}{r_0} = 2$		$\frac{a}{r_0} = 3$		$\frac{a}{r_0} = 4$		$\frac{a}{r_0} = 5$	
Case	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2
1	1.10	0.341	1.26	0.519	1.48	0.672	1.88	0.734	2.17	0.724	2.34	0.704
2	0.66	0.202	1.19	0.491	2.04	0.902	3.34	1.220	4.30	1.300	5.10	1.310
3	0.135	0.00231	0.410	0.0183	1.04	0.0938	2.15	0.293	2.99	0.448	3.69	0.564
4	0.122	0.00343	0.336	0.0313	0.74	0.1250	1.21	0.291	1.45	0.417	1.59	0.492
5	0.090	0.00077	0.273	0.0062	0.71	0.0329	1.54	0.110	2.23	0.179	2.80	0.234
6	0.115	0.00129	0.220	0.0064	0.405	0.0237	0.703	0.062	0.933	0.092	1.13	0.114
7	0.592	0.184	0.976	0.414	1.440	0.664	1.880	0.824	2.08	0.830	2.19	0.813
8	0.227	0.00510	0.428	0.0249	0.753	0.0877	1.205	0.209	1.514	0.293	1.745	0.350
9	0.194	0.00504	0.320	0.0242	0.454	0.0810	0.673	0.172	1.021	0.217	1.305	0.238
10	0.105	0.00199	0.259	0.0139	0.480	0.0575	0.657	0.130	0.710	0.162	0.730	0.175

[a] Poisson's ratio $\nu = 0.30$.